Stochastic trailing string and Langevin dynamics from AdS/CFT

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Abstract: Using the gauge/string duality, we derive a set of Langevin equations describing the dynamics of a relativistic heavy quark moving with constant average speed through the strongly-coupled $\mathcal{N}=4$ SYM plasma at finite temperature. We show that the stochasticity arises at the string world-sheet horizon, and thus is causally disconnected from the black hole horizon in the space-time metric. This hints at the non-thermal nature of the fluctuations, as further supported by the fact that the noise term and the drag force in the Langevin equations do not obey the Einstein relation. We propose a physical picture for the dynamics of the heavy quark in which dissipation and fluctuations are interpreted as medium-induced radiation and the associated quantum-mechanical fluctuations. This picture provides the right parametric estimates for the drag force and the (longitudinal and transverse) momentum broadening coefficients.

Contents

1 Introduction

Motivated by possible strong-coupling aspects in the dynamics of ultrarelativistic heavy ion collisions, there have been many recent applications of the AdS/CFT correspondence to the study of the response of a strongly coupled plasma — typically, that of the $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory at finite temperature — to an external perturbation, so like a "hard probe" — say, a heavy quark, or an electromagnetic current (see the review papers [1–3] for details and more references). Most of these studies focused on the mean field dynamics responsible for dissipation (viscosity, energy loss, structure functions), as encoded in retarded response functions — typically, the 2-point Green's function of the $\mathcal{N}=4$ SYM operator perturbing the plasma. By comparison, the statistical properties of the plasma (in or near thermal equilibrium) have been less investigated. Within the AdS/CFT framework, such investigations would require field quantization in a curved space-time — the $AdS_5 \times S^5$ Schwarzschild geometry dual to the strongly-coupled $\mathcal{N}=4$ SYM plasma — which in general is a very difficult problem. Still, there has been some interesting progress in that sense, which refers to a comparatively simpler problem: that of the quantization of the small fluctuations of the Nambu-Goto string dual to a heavy quark immersed into the plasma.

Several noticeable steps may be associated with this progress: In ref. [4], a prescription was formulated for computing the Schwinger-Keldysh Green's functions at finite temperature within the AdS/CFT correspondence. With this prescription, the quantum thermal distributions are generated via analytic continuation across the horizon singularities in the Kruskal diagram for the AdS_5 Schwarzschild space-time. Using this prescription, one has computed the diffusion coefficient of a non-relativistic heavy quark [5], and the momentum broadening for a relativistic heavy quark which propagates through the plasma at constant

(average) speed [6, 7]. Very recently, in refs. [8, 9], a set of Langevin equations has been constructed which describes the Brownian motion of a non-relativistic heavy quark and of the attached Nambu-Goto string. Within these constructions, the origin of the 'noise' (the random force in the Langevin equations) in the supergravity calculations lies at the black hole horizon, as expected for thermal fluctuations.

The Langevin equations in refs. [8, 9] encompass previous results for the drag force [10, 11] and the diffusion coefficient [5] of a *non-relativistic* heavy quark. But to our knowledge, no attempt has been made so far at deriving corresponding equations for a *relativistic* heavy quark, whose dual description is a trailing string [10, 11]. In fact, the suitability of the Langevin description for the stochastic trailing string was even challenged by the observation that the respective expressions for the drag force and the momentum broadening do not to obey the Einstein relation [7]. The latter is a hallmark of thermal equilibrium and must be satisfied by any Langevin equation describing thermalization. However, Langevin dynamics is more general than thermalization, and as a matter of facts it does apply to the stochastic trailing string, as we will demonstrate in this paper.

Specifically, our objective in what follows is twofold: (i) to show how the Langevin description of the stochastic trailing string unambiguously emerges from the underlying AdS/CFT formalism, and (ii) to clarify the physical interpretation of the associated noise term, in particular, its non-thermal nature.

Our main conclusion is that the stochastic dynamics of the relativistic quark is fundamentally different from the Brownian motion of a non-relativistic quark subjected to a thermal noise. Within the supergravity calculation, this difference manifests itself via the emergence of an event horizon on the string world-sheet [6, 7], which lies in between the Minkowski boundary and the black hole horizon, and which governs the stochastic dynamics of the fast moving quark. With our choice for the radial coordinate z in AdS_5, the Minkowski boundary lies at $z = 0$, the black hole horizon at $z_H = 1/T$, and the world-sheet horizon at $z_s = z_H/\sqrt{\gamma}$, where $\gamma = 1/\sqrt{1 - v^2}$ is the Lorentz factor of the heavy quark. (We assume that the quark is pulled by an external force in such a way that its average velocity remains constant.) The presence of the world-sheet horizon means that the dynamics of the upper part of the string at $z < z_s$ (including the heavy quark at $z \simeq 0$) is causally disconnected from that of its lower part at $z_s < z < z_H$, and thus cannot be influenced by thermal fluctuations originating at the black hole horizon.

This conclusion is supported by the previous calculations of the momentum broadening for the heavy quark [6, 7], which show that the relevant correlations are generated (via analytic continuation in the Kruskal plane) at the *world-sheet* horizon, and not at the black hole one. Formally, these correlations look as being thermal (they involve the Bose-Einstein distribution), but with an effective temperature $T_{\text{eff}} = T/\sqrt{\gamma}$, which is the Hawking temperature associated to the world-sheet horizon. Thus, no surprisingly, our explicit construction of the Langevin equations will reveal that the corresponding noise terms arise from this world-sheet horizon.

The Langevin equations for the relativistic heavy quark will be constructed in two different ways: (1) by integrating out the quantum fluctuations of the upper part of the string, from the world-sheet horizon up to the boundary, and (2) by integrating out the

string fluctuations only within an infinitesimal strip in z, from the world-sheet horizon at $z = z_s$ up to the 'stretched' horizon at $z = z_s(1 - \epsilon)$ with $\epsilon \ll 1$; this generates a 'bulk' noise term at the stretched horizon, whose effects then propagate upwards the string, via the corresponding classical solutions. Both procedures provide exactly the same set of Langevin equations, which encompass the previous results for the drag force [10, 11] and for the (longitudinal and transverse) momentum broadening [6, 7]. In these manipulations, the lower part of the string at $z > z_s$ and, in particular, the black hole horizon, do not play any role, as expected from the previous argument on causality.

If the relevant fluctuations are not of thermal nature, then why do they *look* as being thermal? What is their actual physical origin? And what is the role played by the thermal bath? To try and answer such questions, we will rely on a physical picture for the interactions between an energetic parton and the strongly-coupled plasma which was proposed in refs. [2, 12–14], and that we shall here more specifically develop for the problem at hand. In this picture, both the energy loss ('drag force') and the momentum broadening ('noise term') are due to medium-induced radiation. This is reminiscent of the mechanism of energy loss of a heavy, or light, quark at weak coupling [15–18], with the main difference being in the cause of the medium-induced radiation. At weak coupling, multiple scattering off the plasma constituents frees gluonic fluctuations in the quark wavefunction, while at strong coupling the plasma exerts a force, proportional to T^2, acting to free quanta from the heavy quark as radiation. In the gravity description, this appears as a force pulling energy in the trailing string towards the horizon. At either weak or strong coupling, quanta are freed when their virtuality is smaller than a critical value, the *saturation momentum* Q_s; at strong coupling and for a fast moving quark, this scales like $Q_s \sim \sqrt{\gamma}T$. Within this picture, the world-sheet horizon at $z_s \sim 1/Q_s$ corresponds to the causal separation between the highly virtual quanta ($Q \gg Q_s$), which cannot decay into the plasma and thus are a part of the heavy quark wavefunction, and the low virtuality ones, with $Q \lesssim Q_s$, which have already been freed, thus causing energy loss. The recoil of the heavy quark due to the random emission of quanta with $Q \lesssim Q_s$ is then responsible for its momentum broadening.

From his perspective, the noise terms in the Langevin equations for the fast moving quark reflect quantum fluctuations in the emission process. Of course, the presence of the surrounding plasma is essential for this emission to be possible in the first place (a heavy quark moving at constant speed through the vacuum could not radiate), but the plasma acts merely as a background field, which acts towards reducing the virtuality of the emitted quanta and thus allows them to decay. The genuine thermal fluctuations on the plasma are unimportant when $\gamma \gg 1$, although when $\gamma \simeq 1$ they are certainly the main source of stochasticity, as shown in [8, 9]. Besides, we see no role for Hawking radiation of supergravity quanta at any value of γ.

This picture is further corroborated by the study of a different physical problem, where the thermal effects are obviously absent, yet the mathematical treatment within AdS/CFT is very similar to that for the problem at hand: this is the problem of a heavy quark propagating with constant acceleration a through the vacuum of the strongly-coupled $\mathcal{N} = 4$ SYM theory [14, 19, 20]. The accelerated particle can radiate, and this radiation manifests itself through the emergence of a world-sheet horizon, leading to dissipation

and momentum broadening. The fluctuations generated at this horizon are once again thermally distributed, with an effective temperature $T_{\rm eff} = a/2\pi$. In that context, it is natural to interpret the induced horizon as the AdS dual of the Unruh effect [21, 22]: the accelerated observer perceives the Minkowski vacuum as a thermal state with temperature $a/2\pi$. For an inertial observer, this is interpreted as follows [23]: the accelerated particle can radiate and the correlations induced by the backreaction to this radiation are such that the excited states of the emitted particle are populated according to a thermal distribution. Most likely, a similar interpretation holds also for the thermal-like correlations generated at the world-sheet horizon in the problem at hand — that of a relativistic quark propagating at constant speed through a thermal bath. It would be interesting to identify similar features in other problems which exhibit accelerated motion, or medium-induced radiation, or both, so like the rotating string problem considered in ref. [24].

The paper is organized as follows: In section 2 we construct the Langevin equations describing the stochastic dynamics of the string endpoint on the boundary of AdS_5, i.e., of the relativistic heavy quark. Our key observation is that, in the Kruskal-Keldysh quantization of the small fluctuations of the trailing string, the stochasticity is generated exclusively at the world-sheet horizon. This conclusion is further substantiated by the analysis in section 3 where we follow the progression of the fluctuations along the string, from the world-sheet horizon up to the string endpoint on the boundary. We thus demonstrate that the noise correlations are faithfully transmitted from the stretched horizon to the heavy quark, via the fluctuations of the string. Finally, section 4 contains our physical discussion. First, in section 4.1, we argue that the Langevin equations do not describe thermalization, although they do generate thermal-like momentum distributions, but at a fictitious temperature which is not the same as the temperature of the plasma, and is moreover different for longitudinal and transverse fluctuations. Then, in section 4.2, we develop our physical picture for medium-induced radiation and parton branching, which emphasizes the quantum-mechanical nature of the stochasticity.

2   Boundary picture of the stochastic motion

In this section we will construct a set of Langevin equations for the stochastic dynamics of a relativistic heavy quark which propagates with uniform average velocity through a strongly-coupled $\mathcal{N} = 4$ SYM plasma at temperature T. To that aim, we will follow the general strategy in ref. [9], that we will extend to a fast moving quark and the associated trailing string. In this procedure, we will also rely on previous results in the literature [6, 7] concerning the classical solutions for the fluctuations of the trailing string and their quantization via analytic continuation in the Kruskal plane.

2.1   The trailing string and its small fluctuations

The AdS dual of the heavy quark is a string hanging down in the radial direction of AdS_5, with an endpoint (representing the heavy quark) attached to a D7-brane whose radial coordinate fixes the bare mass of the quark. The string dynamics is encoded in the

Nambu-Goto action,

$$S \,=\, -\frac{1}{2\pi\ell_s^2}\int \mathrm{d}^2\sigma\sqrt{-\det h_{\alpha\beta}}\,, \qquad\qquad h_{\alpha\beta} \,=\, g_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu\,, \qquad (2.1)$$

where σ^α, $\alpha = 1,\,2$, are coordinates on the string world-sheet, $h_{\alpha\beta}$ is the induced world-sheet metric, and $g_{\mu\nu}$ is the metric of the AdS_5-Schwarzschild space-time, chosen as

$$\mathrm{d}s^2 = \frac{R^2}{z_H^2 z^2}\left(-f(z)\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2 + \frac{\mathrm{d}z^2}{f(z)}\right), \qquad (2.2)$$

where $f(z) = 1 - z^4$ and $T = 1/\pi z_H$ is the Hawking temperature. (As compared to the introduction, we have switched to a dimensionless radial coordinate.) The position z_m of the D7-brane on which ends the string is related to the heavy quark mass M_Q as $z_m = \sqrt{\lambda}T/2M_Q$. In what follows we shall assume the quark to be heavy enough for $z_m \ll 1$.

The quark is moving along the longitudinal axis x^3 with constant (average) velocity v in the plasma rest frame. For this to be possible, the quark must be subjected to some external force, which compensates for the energy loss towards the plasma. The profile of the string corresponding to this steady (average) motion is known as the 'trailing string'. This is obtained by solving the equations of motion derived from eq. (2.1) with appropriate boundary conditions, and reads [10, 11]

$$x_0^3 \,=\, vt + \frac{vz_H}{2}\left(\arctan z - \operatorname{arctanh}z\right). \qquad (2.3)$$

In what follows we shall be interested in small fluctuations around this steady solution, which can be either longitudinal or transverse: $x^3 = x_0^3 + \delta x_\ell(t,z)$ and $x_\perp = \delta x_\perp(t,z)$. To quadratic order in the fluctuations, the Nambu-Goto action is then expanded as (in the static gauge $\sigma^\alpha = (t,z)$)

$$S = -\frac{\sqrt{\lambda}T z_s^2}{2}\int \mathrm{d}t\mathrm{d}z\frac{1}{z^2} + \int \mathrm{d}t\mathrm{d}z P^\alpha \partial_\alpha \delta x_\ell$$
$$-\frac{1}{2}\int \mathrm{d}t\mathrm{d}z\left[T_\ell^{\alpha\beta}\partial_\alpha \delta x_\ell\,\partial_\beta \delta x_\ell + T_\perp^{\alpha\beta}\partial_\alpha \delta x_\perp^i\,\partial_\beta \delta x_\perp^i\right], \quad (2.4)$$

where $z_s \equiv \sqrt[4]{1-v^2} = 1/\sqrt{\gamma}$ and[1] [7]

$$P^\alpha \,=\, \frac{\pi v\sqrt{\lambda}T^2}{2z_s^2}\begin{pmatrix} \frac{z_H}{z^2(1-z^4)} \\ 1 \end{pmatrix}, \qquad (2.5)$$

$$T_\perp^{\alpha\beta} = z_s^4 T_\ell^{\alpha\beta} = -\frac{\pi\sqrt{\lambda}T^2}{2z_s^2}\begin{pmatrix} \frac{z_H}{z^2}\frac{1-(zz_s)^4}{(1-z^4)^2} & \frac{v^2}{1-z^4} \\ \frac{v^2}{1-z^4} & \frac{z^4-z_s^4}{z_H z^2} \end{pmatrix}. \qquad (2.6)$$

The quantities $T_{\ell,\perp}^{\alpha\beta}$ have the meaning of local stress tensors on the string. At high energy, the components $T_\ell^{\alpha\beta}$ of the longitudinal stress tensor are parametrically larger, by a factor $\gamma^2 \gg 1$, than the corresponding components $T_\perp^{\alpha\beta}$ of the transverse stress tensor. This difference reflects the strong energy-dependence of the gravitational interactions.

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[1]In eq. (2.6), we have corrected an overall sign error in eq. (22) of ref. [7].

Using $\partial_\alpha P^\alpha = 0$, one sees that the term linear in the fluctuations in eq. (2.4) does not affect the equations of motion, which therefore read

$$\partial_\alpha(T^{\alpha\beta}\partial_\beta\psi) = 0, \qquad \psi = \delta x_\ell, \ \delta x_\perp, \qquad (2.7)$$

in compact notations which treat on the same footing the longitudinal and transverse fluctuations. Upon expanding in Fourier modes,

$$\psi(t, z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \psi(\omega, z) \, e^{-i\omega t}, \qquad (2.8)$$

this yields

$$\left\{a(z)\partial_z^2 - 2b(\omega, z)\partial_z + c(\omega, z)\right\}\psi(\omega, z) = 0, \qquad (2.9)$$

where

$$
\begin{aligned}
a(z) &= z(1 - z^4)^2(z_s^4 - z^4), \\
b(\omega, z) &= (1 - z^4)\left[1 - z^8 - v^2(1 - z^2 + i\omega z_H z^3)\right], \\
c(\omega, z) &= \omega z_H z\left[\omega z_H(1 - z^4) + v^2 z^4(\omega z_H + 4iz)\right].
\end{aligned}
\qquad (2.10)
$$

The zeroes of $a(z)$ determine the regular singular points of this equation. In particular, the special role played by the point z_s as a world-sheet horizon becomes manifest at this level: for $z = z_s$ the value of $\partial_z\psi(\omega, z_s)$ is determined from the equation of motion. This means that fluctuations of the string at $z < z_s$ are causally disconnected from those below the location of the world-sheet horizon. Clearly, this world-sheet horizon must lie below the string endpoint on the D7-brane, that is $z_s > z_m$. This condition can be rewritten as

$$\gamma < \frac{4M_Q^2}{\lambda T^2} = \frac{1}{z_m^2}, \qquad (2.11)$$

which is recognized as a limitation on the energy (or velocity) of the heavy quark up to which the present formalism applies. Note that the term in the r.h.s. of this inequality is a pure number (independent of λ), which can be made arbitrarily large by taking the mass of the heavy quark large enough.

2.2 Keldysh Green function in AdS/CFT

In what follows we construct solutions to eqs. (2.9)–(2.10) for the string fluctuations which are well defined everywhere in the Kruskal diagram for the AdS_5 Schwarzschild space-time (see figure 1). These solutions are uniquely determined by their boundary conditions at the two Minkowski boundaries — in the right (R) and, respectively, left (L) quadrants of the Kruskal diagram — together with the appropriate conditions of analyticity in the Kruskal variables U and V (as explained in [4]). The latter amount to quantization prescriptions which impose infalling conditions on the positive-frequency modes and outgoing conditions on the negative-frequency ones. These prescriptions ultimately generate the quantum Green's functions at finite-temperature and in real time, which are time-ordered
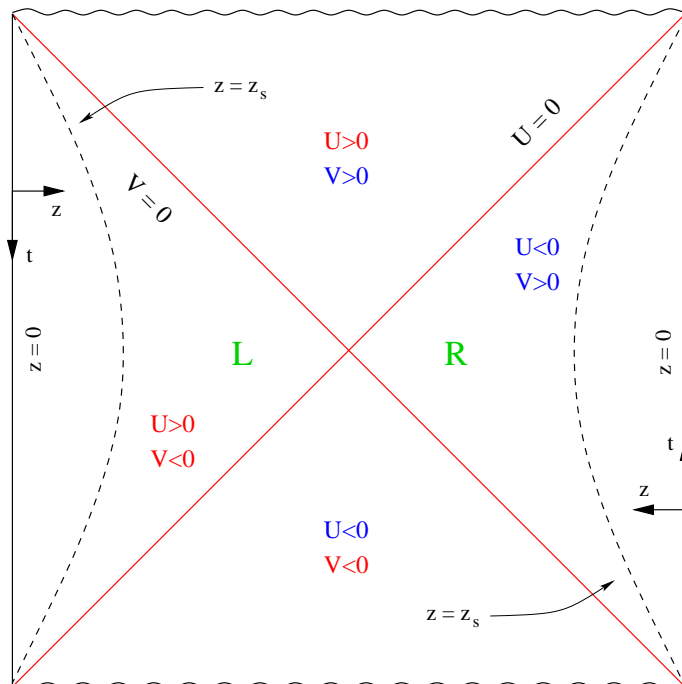
Figure 1. Kruskal diagram for AdS_5 Schwarzschild metric. The Kruskal coordinates U and V are defined by $UV = \frac{z-1}{z+1}\,\mathrm{e}^{-2\arctan(z)}$, $V/U = -\mathrm{e}^{4\pi T t}$. Hence, the black horizon at $z = 1$ corresponds to $UV = 0$, whereas $V = 0$ ($U = 0$) corresponds to $\mathrm{Re}\,t \to -\infty$ ($\mathrm{Re}\,t \to \infty$). The position of the induced world-sheet horizon is shown with dashed lines in both R and L quadrants.

along the Keldysh contour [4]. Specifically, the time variables on the R and, respectively, L boundary in figure 1 correspond to the chronological and, respectively, antichronological branches of the Keldysh time contour.

As usual in the framework of AdS/CFT, we are interested in the classical action expressed as a functional of the fields on the boundary. We denote by $\psi_R(t_R, z)$ and $\psi_L(t_R, z)$ the classical solutions in the R and L quadrant, respectively. Making use of the equations of motion and integrating by parts, the classical action reduces to its value on the boundary of the Kruskal diagram, i.e., the R and L Minkowski boundaries:

$$
S_{\mathrm{bndry}} = \int \mathrm{d}t_R \left[-P^z \psi_R + \frac{1}{2}\,\psi_R T^{z\beta} \partial_\beta \psi_R \right]_{z_R = z_m} - \int \mathrm{d}t_L \left[-P^z \psi_L + \frac{1}{2}\,\psi_L T^{z\beta} \partial_\beta \psi_L \right]_{z_L = z_m},
\tag{2.12}
$$

where it is understood that the terms involving P_z exist only in the longitudinal sector. We recall that $z_m \ll 1$ is the radial location of the string endpoint on the D7-brane.

In eq. (2.12), the world-sheet index β can take *a priori* both values t and z, but the contribution corresponding to $\beta = t$ is in fact zero, since the respective integrand is an odd function of t. This is worth noticing since in ref. [7] it was found that the dominant contribution to the imaginary part of the retarded propagator at high energy ($\gamma \gg 1$) comes from the piece proportional to T^{zt}. We will later see how that contribution arises in the present calculation, where only the piece proportional to T^{zz} survives in eq. (2.12).

Switching to the frequency representation, we introduce a basis of retarded and advanced solutions, $\psi_{\mathrm{ret}}(\omega, z)$ and $\psi_{\mathrm{adv}}(\omega, z)$, which are normalized such that $\psi_{\mathrm{ret}}(\omega, 0) = \psi_{\mathrm{adv}}(\omega, 0) = 1$. They obey $\psi_{\mathrm{ret}}(\omega, z) = \psi_{\mathrm{ret}}^*(-\omega, z)$, and similarly for ψ_{adv}. These solutions are truly boundary-to-bulk propagators in Fourier space. They have been constructed in ref. [7] (see also [6]) from which we quote the relevant results.

Note first that, unlike what happens for a static (or non-relativistic) quark [9], the retarded and advanced solutions are not simply related to each other by complex conjugation: one rather has

$$\psi_{\mathrm{adv}}(\omega, z) \,=\, [g(z)]^{i\omega/2} \, [g(z/z_s)]^{-i\omega z_H/2z_s} \, \psi_{\mathrm{ret}}^*(\omega, z), \tag{2.13}$$

with $g(z) = \frac{1+z}{1-z} \, \mathrm{e}^{-2\arctan z}$. Near the boundary ($z \ll 1$), these solutions behave as follows

$$\psi_{\mathrm{ret}}(\omega, z) \,=\, \left(1 + \frac{z_H^2 \omega^2}{2 z_s^4} z^2 + \mathcal{O}(z^4)\right) + C_{\mathrm{ret}}(\omega)\left(z^3 + \mathcal{O}(z^5)\right) \tag{2.14}$$

$$\psi_{\mathrm{adv}}(\omega, z) \,=\, \left(1 + \frac{z_H^2 \omega^2}{2 z_s^4} z^2 + \mathcal{O}(z^4)\right) + C_{\mathrm{adv}}(\omega)\left(z^3 + \mathcal{O}(z^5)\right), \tag{2.15}$$

where the expansion involving even (odd) powers of z is that of the non-normalizable (normalizable) mode. The coefficients of the normalizable mode are related by

$$C_{\mathrm{adv}}(\omega) = C_{\mathrm{ret}}^*(\omega) - iX(\omega), \tag{2.16}$$

with

$$X(\omega) \,=\, \frac{2\omega z_H}{3} \, v^2 \gamma^2, \qquad \mathrm{Im}\, C_{\mathrm{ret}}(\omega) \,=\, i\frac{\omega z_H}{3}. \tag{2.17}$$

The real part of coefficient $C_{\mathrm{ret}}(\omega)$ has been numerically evaluated in ref. [7]. Here, we only need to know that, at small frequency $\omega \ll z_s T$, its real part is comparatively smaller: $\mathrm{Re}\, C_{\mathrm{ret}}(\omega) \sim \mathcal{O}(\omega^2 z_H^2 / z_s^2)$. Note that, at high energy ($\gamma \gg 1$), X dominates over $\mathrm{Im}\, C_{\mathrm{ret}}$ in eq. (2.16).

Consider also the approach towards the world-sheet horizon ($z = z_s$) from the above ($z < z_s$): there, ψ_{ret} remains regular, whereas ψ_{adv} has a branching point:

$$\psi_{\mathrm{adv}}(\omega, z) \,\propto\, (z_s - z)^{\frac{i\omega z_H}{2z_s}} [1 + \mathcal{O}(z_s - z)] \,. \tag{2.18}$$

This shows that $\Psi_{\mathrm{adv}}(\omega, t, z) \equiv \mathrm{e}^{-i\omega t} \psi_{\mathrm{adv}}(\omega, z)$ is an outgoing wave: with increasing time, the phase remains constant while departing from the horizon. One can similarly argue that $\Psi_{\mathrm{ret}}(\omega, t, z) = \mathrm{e}^{-i\omega t} \psi_{\mathrm{ret}}(\omega, z)$ is an infalling solution [7].

We now expand the general solution in the right and left quadrants of the Kruskal diagram in this retarded/advanced basis:

$$\psi_R(\omega, z) = A(\omega)\psi_{\mathrm{ret}}(\omega, z) + B(\omega)\psi_{\mathrm{adv}}(\omega, z),$$
$$\psi_L(\omega, z) = C(\omega)\psi_{\mathrm{ret}}(\omega, z) + D(\omega)\psi_{\mathrm{adv}}(\omega, z). \tag{2.19}$$

We need four conditions to determine the four unknown coefficients A, B, C, and D. Two of them are provided by the boundary values at the R and L Minkowski boundaries, that

we denote as $\psi_R^0(\omega)$ and $\psi_L^0(\omega)$, respectively. The other two are determined by analyticity conditions in the Kruskal plane, which allows one to connect the solution in the L quadrant to that in the R quadrant. With reference to figure 1, one sees that this requires crossing two types of horizons: the world-sheet horizons in both R and L quadrants, and the black hole horizons at $U = 0$ and $V = 0$. The detailed matching at these horizons, following the prescription of ref. [4], has been performed in appendix B of ref. [7], with the following results: the multiplicative factors associated with crossing the black hole horizons precisely compensate each other, unlike those associated with crossing the world-sheet horizons, which rather enhance each other. Hence, the net result comes from the world-sheet horizons alone,[2] and reads

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\frac{\omega}{z_s T}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{2.20}$$

The two independent coefficients can now be determined from the boundary values $\psi_R^0(\omega)$ and $\psi_L^0(\omega)$. This eventually yields

$$A(\omega) = (1 + n)(\omega)\psi_R^0(\omega) - n(\omega)\psi_L^0(\omega), \tag{2.21}$$
$$B(\omega) = n(\omega)\left[\psi_L^0(\omega) - \psi_R^0(\omega)\right]. \tag{2.22}$$

Here $n(\omega) = 1/(\mathrm{e}^{\omega/z_s T} - 1)$ is the Bose-Einstein thermal distribution with the effective temperature $T_{\mathrm{eff}} = z_s T = T/\sqrt{\gamma}$. As it should be clear from the previous manipulations, this effective thermal distribution has been generated via the matching conditions at the R and L world-sheet horizons, cf. eq. (2.20).

The equations simplify if one introduces 'average' (or 'classical') and 'fluctuating' variables, according to $\psi_r \equiv (\psi_R + \psi_L)/2$ and $\psi_a \equiv \psi_R - \psi_L$, and similarly for the boundary values. One then finds

$$\psi_r(\omega, z) = \psi_r^0(\omega)\psi_{\mathrm{ret}}(\omega, z) + \frac{1 + 2n(\omega)}{2}\psi_a^0(\omega)(\psi_{\mathrm{ret}}(\omega, z) - \psi_{\mathrm{adv}}(\omega, z)),$$
$$\psi_a(\omega, z) = \psi_a^0(\omega)\psi_{\mathrm{adv}}(\omega, z), \tag{2.23}$$

and the boundary action takes a particularly simple form:

$$S_{\mathrm{bndry}} = \frac{1}{2} \int \frac{\mathrm{d}\omega}{2\pi} T^{zz}(z)\left[\psi_r(-\omega, z)\partial_z\psi_a(\omega, z) + (r \leftrightarrow a)\right]_{z=z_m}. \tag{2.24}$$

(We have here omitted the term linear in the fluctuations, since this does not matter for the calculation of the 2-point Green's functions. This term will be reinserted in the next subsection.) By combining the above equations, we finally deduce

$$iS_{\mathrm{bndry}} = -i \int \frac{\mathrm{d}\omega}{2\pi} \psi_a^0(-\omega)G_R^0(\omega)\psi_r^0(\omega) - \frac{1}{2} \int \frac{\mathrm{d}\omega}{2\pi} \psi_a^0(-\omega)G_{\mathrm{sym}}(\omega)\psi_a^0(\omega), \tag{2.25}$$

[2]This point is even more explicit in the analysis in ref. [6], where a different set of coordinates was used, in which the world-sheet metric is diagonal. With those coordinates, the only horizons to be crossed when going from the R to the L boundary in the respective Kruskal diagram are the *world-sheet* horizons.

with the retarded and symmetric Green's functions defined as

$$G^0_{\perp,\,R}(\omega) = z_s^4 G^0_{\ell,\,R}(\omega) \,=\, -\frac{1}{2}\, T^{zz}_\perp(z)\, \partial_z \Big[\psi_{\mathrm{ret}}(\omega,z)\psi_{\mathrm{adv}}(-\omega,z)\Big]_{z=z_m}, \qquad (2.26)$$

and, respectively,

$$G_{\mathrm{sym}}(\omega) = -(1 + 2n(\omega))\,\mathrm{Im}\, G^0_R(\omega). \qquad (2.27)$$

Note that eq. (2.27) is formally the same as the fluctuation-dissipation theorem (or 'KMS condition') characteristic of thermal equilibrium, but with an effective temperature $T_{\mathrm{eff}} = z_s T$. By also using eqs. (2.14)–(2.15) together with the expression of $T^{zz}_\perp$ given in (2.6), one finally deduces

$$G^0_{\perp,\,R}(\omega) \,=\, z_s^4 G^0_{\ell,\,R} \,=\, G_R(\omega) - \gamma M_Q \omega^2\,, \qquad (2.28)$$

where

$$G_R(\omega) \equiv -Y\big(C_{\mathrm{ret}}(\omega) + C^*_{\mathrm{adv}}(\omega)\big),$$
$$Y \equiv \frac{3\sqrt{\lambda}}{4\pi\gamma z_H^3}\,, \qquad M_Q \,=\, \frac{\sqrt{\lambda}T}{2z_m} \,=\, \frac{\sqrt{\lambda}\,r_m}{2\pi R^2}\,. \qquad (2.29)$$

M_Q is the (bare) rest mass of the heavy quark and is independent of temperature, as manifest in his last rewriting. (r_m denotes the position of the D7-brane in the usual radial coordinate r, which is related to z as $z/\pi T = R^2/r$.) At finite temperature, this mass receives thermal corrections, as encoded in the contribution of $\mathcal{O}(\omega^2)$ to $\mathrm{Re}\, C_{\mathrm{ret}}(\omega)$; such corrections are however negligible at high energy, since their contribution to $G_R(\omega)$ is not enhanced by a factor of γ (unlike M_Q).

Note that the previous formulae fully specify the imaginary part of the retarded propagator, and hence also $G_{\mathrm{sym}}(\omega)$. Namely, by using (cf. eqs. (2.16)–(2.17))

$$\mathrm{Im}\, C_{\mathrm{ret}}(\omega) + \mathrm{Im}\, C^*_{\mathrm{adv}}(\omega) \,=\, \frac{2\omega z_H}{3}(1 + v^2\gamma^2) \,=\, \frac{2\omega z_H}{3}\,\gamma^2, \qquad (2.30)$$

one immediately finds

$$\mathrm{Im}\, G_R(\omega) \,=\, -\omega\gamma\eta\,, \qquad \text{with} \qquad \eta \equiv \frac{\pi\sqrt{\lambda}}{2}\,T^2\,. \qquad (2.31)$$

Remarkably, this exact result involves just a term linear in ω. On the other hand, we expect $\mathrm{Re}\, G_R(\omega)$ to receive contributions to all orders in ω starting at $\mathcal{O}(\omega^2)$.

The above expression for G^0_R, eq. (2.28), coincides with that originally derived in ref. [7], although the definition used there for the retarded propagator was different, namely

$$G^0_R(\omega) \,\equiv\, -\Psi^*_{\mathrm{ret}} T^{z\beta}\partial_\beta\Psi_{\mathrm{ret}}|_{z=z_m}\,. \qquad (2.32)$$

(Recall that $\Psi_{\mathrm{ret}}(\omega,t,z) = \mathrm{e}^{-i\omega t}\psi_{\mathrm{ret}}(\omega,z)$.) With this definition, the dominant contribution to the imaginary part at high energy — the term proportional to $X(\omega)$ — arises from the time derivative of the retarded solution. Although it does not naturally emerge

when constructing the boundary action in the Kruskal plane, this formula (2.32) has another virtue, which will be useful later on: with this definition, the imaginary part of the retarded propagator,

$$\text{Im}\, G_R^0(\omega) \;=\; \frac{1}{2i}\, T^{z\beta}\Big(\Psi_{\text{ret}}^*\partial_\beta\Psi_{\text{ret}} - \Psi_{\text{ret}}\partial_\beta\Psi_{\text{ret}}^*\Big)\,, \qquad (2.33)$$

can be evaluated at any z, since the r.h.s. of eq. (2.33) is independent of z. Indeed, as noticed in ref. [7], the world-sheet current

$$J^\alpha \;=\; \frac{1}{2i}\, T^{\alpha\beta}\Big(\Psi_{\text{sol}}^*\partial_\beta\Psi_{\text{sol}} - \Psi_{\text{sol}}\partial_\beta\Psi_{\text{sol}}^*\Big)\,, \qquad (2.34)$$

($\Psi_{\text{sol}}(t,z)$ is an arbitrary solution to the classical EOM (2.7)) is conserved by the equations of motion, $\partial_\alpha J^\alpha = 0$. When $\Psi_{\text{sol}}(t,z) = \Psi_{\text{ret}}(\omega,t,z)$, this conservation law reduces to $\partial_z J^z = 0$. As we shall shortly see, $\text{Im}\, G_R^0(\omega)$ is a measure of the energy loss of the heavy quark towards the plasma. Thus the fact this quantity is independent of z is a statement about the conservation of the energy flux down the string in the present, steady, situation.

2.3 A Langevin equation for the heavy quark

Following the general strategy of AdS/CFT, the boundary action (2.25) can be used to generate the correlation functions of the $\mathcal{N} = 4$ SYM operator which couples to the boundary value of the field — in this case, the Schwinger-Keldysh 2-point functions of the force operator acting on the heavy quark [4, 6, 7, 9]. Alternatively, in what follows, this action will be used to derive stochastic equations for the string endpoint, in the spirit of the Feynman-Vernon 'influence functional' [25] (see also ref. [9]).

To that aim we start with the following path integral which encodes the (quantum and thermal) dynamics of the string fluctuations in the Gaussian approximation of interest:

$$Z = \int \big[D\psi_R^0 D\psi_R\big]\big[D\psi_L^0 D\psi_L\big]\, e^{iS_R - iS_L}\,. \qquad (2.35)$$

This involves two types of functional integrations: (i) those with measure $[D\psi_R D\psi_L]$, which run over all the string configurations $\psi_{R,L}(t,z)$ (in the corresponding quadrants of the Kruskal diagram) with given boundary values $\psi_{R,L}^0(t)$, and (ii) those with measure $[D\psi_R^0 D\psi_L^0]$, which run over all the possible paths $\psi_{R,L}^0(t)$ for these endpoint values.

Performing the Gaussian path integral over the bulk configurations amounts to evaluating the action in the exponent of eq. (2.35) with the classical solutions computed in the previous section. This leaves us with the boundary action in eq. (2.25), which determines the dynamics of the string endpoints — i.e., of the heavy quark — and which is itself Gaussian. To perform the corresponding path integral it is convenient to 'break' the quadratic term for the fluctuating fields ψ_a^0, by introducing an auxiliary stochastic field $\xi(t)$. Then the partition function becomes

$$Z = \int \big[D\psi_r^0\big]\big[D\psi_a^0\big][D\xi]\; e^{-\int \mathrm{d}t\mathrm{d}t'\,\frac{1}{2}\big[\xi(t)G_{\text{sym}}^{-1}(t,t')\xi(t')\big]}$$

$$\exp\left\{-i\int \mathrm{d}t\mathrm{d}t'\,\psi_a^0(t)\big[G_R^0(t,t')\psi_r^0(t') + \delta(t-t')\big(P^z - \xi(t')\big)\big]\right\},\quad (2.36)$$

where we recall that the term involving P^z appears only in the longitudinal sector. The integral over ψ_a acts as a constraint which enforces a Langevin equation for the 'average' field ψ_r. This equation reads

$$\int \mathrm{d}t' G_R^0(t,t')\psi_r^0(t') + P^z - \xi(t) = 0, \qquad \langle \xi(t)\xi(t')\rangle = G_{\mathrm{sym}}(t,t'), \qquad (2.37)$$

and is generally non-local in time. At this point it is convenient to focus on the large time behaviour, as controlled by the small-frequency expansion of the Green's functions G_R and G_{sym}, and also distinguish between longitudinal and transverse fluctuations. As discussed in section 2.2, for $\omega \ll z_s T$, the retarded propagator reduces to its imaginary part, eq. (2.31) (in addition to the bare mass term). In the same limit, one can use $1 + 2n(\omega) \simeq 2z_s T/\omega$ to simplify the expression of $G_{\mathrm{sym}}(\omega)$, which then becomes independent of ω:

$$\begin{aligned}
G_{\perp,\,\mathrm{sym}}(\omega) &\simeq \pi\sqrt{\lambda}\,\gamma^{1/2}T^3 \equiv \kappa_\perp, \\
G_{\ell,\,\mathrm{sym}}(\omega) &\simeq \pi\sqrt{\lambda}\,\gamma^{5/2}T^3 \equiv \kappa_\ell.
\end{aligned} \qquad (2.38)$$

This in turn implies that, when probed over large time separations $t - t' \gg 1/z_s T$, the retarded propagator can be replaced by a local time derivative ('friction force'), while the noise-noise correlator looks local in time ('white noise'). The we can write

$$\gamma M_Q \frac{\mathrm{d}^2\delta x_\perp}{\mathrm{d}t^2} = -\gamma\eta\frac{\mathrm{d}\delta x_\perp}{\mathrm{d}t} + \xi_\perp(t), \qquad \langle \xi_\perp(t)\xi_\perp(t')\rangle = \kappa_\perp\delta(t-t'), \qquad (2.39)$$

for the transverse modes and, respectively (note that $P^z = \gamma\eta v$),

$$\gamma^3 M_Q \frac{\mathrm{d}^2\delta x_\ell}{\mathrm{d}t^2} = -\gamma^3\eta\frac{\mathrm{d}\delta x_\ell}{\mathrm{d}t} - \gamma\eta v + \xi_\ell(t), \qquad \langle \xi_\ell(t)\xi_\ell(t')\rangle = \kappa_\ell\,\delta(t-t'), \quad (2.40)$$

for the longitudinal one. The physical interpretation of these equations becomes more transparent if they are first rewritten in terms of the respective momenta $p_\perp = \gamma M_Q v_\perp$ and $p_\ell = \gamma M_Q v_\ell$, with $v_\perp = \mathrm{d}\delta x_\perp/\mathrm{d}t$ and $v_\ell = v + \mathrm{d}\delta x_\ell/\mathrm{d}t$.

At this point, we come across a rather subtle point: in all the equations written so far, the Lorentz factor γ is evaluated with the *average* velocity v of the heavy quark — the one which enters the trailing string solution (2.3). However, the event-by-event fluctuations of the velocity turn out to be significantly large (especially in the longitudinal sector; see below), and then it becomes appropriate to define the *event-by-event* (or 'fluctuating') momenta $p_\perp$ and p_ℓ by using the respective, event-by-event, Lorentz factor, as evaluated with the instantaneous velocity. For more clarity, let us temporarily denote by v_0 and γ_0 the average velocity and the associated Lorentz factor, $\gamma_0 \equiv 1/\sqrt{1-v_0^2}$, and reserve the notations v and γ for the respective fluctuating quantities:

$$v^2 = v_\ell^2 + v_\perp^2 = \left(v_0 + \frac{\mathrm{d}\delta x_\ell}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\delta x_\perp}{\mathrm{d}t}\right)^2, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}. \qquad (2.41)$$

When taking the time derivatives of $p_\perp$ and p_ℓ, as associated with variations in $v_\perp$ and, respectively, v_ℓ, one must also take into account the corresponding change in the γ-factor.

Consider the longitudinal sector first:

$$\frac{1}{M_Q}\frac{\mathrm{d}p_\ell}{\mathrm{d}t} = \left(\gamma + v_\ell\frac{\partial\gamma}{\partial v_\ell}\right)\frac{\mathrm{d}v_\ell}{\mathrm{d}t} = \left(\gamma + v_\ell^2\gamma^3\right)\frac{\mathrm{d}^2\delta x_\ell}{\mathrm{d}t^2}$$
$$= \gamma^3\frac{\mathrm{d}^2\delta x_\ell}{\mathrm{d}t^2} \simeq \gamma_0^3\frac{\mathrm{d}^2\delta x_\ell}{\mathrm{d}t^2}\,, \tag{2.42}$$

where the last, approximate, equality follows since the fluctuations are comparatively small, hence $v_\ell \simeq v_0$ and $\gamma \simeq \gamma_0$. The final result above is recognized as the expression in the l.h.s. of eq. (2.40). To the same accuracy, we can write (with $\delta v_\ell = \mathrm{d}\delta x_\ell/\mathrm{d}t$)

$$\gamma v_\ell \simeq \left(\gamma_0 + \frac{\partial\gamma}{\partial v_\ell}\delta v_\ell\right)(v_0 + \delta v_\ell) \simeq \gamma_0 v_0 + (\gamma_0 + \gamma_0^3 v_0^2)\delta v_\ell = \gamma_0 v_0 + \gamma_0^3\delta v_\ell\,, \tag{2.43}$$

in which we recognize the terms multiplying η in the r.h.s. of eq. (2.40). Notice the remarkable fact that the variation of the longitudinal momentum for the relativistic heavy quark is proportional to γ^3, and not to γ.

Consider similarly the transverse sector. The analog of eq. (2.42) reads

$$\frac{1}{M_Q}\frac{\mathrm{d}p_\perp}{\mathrm{d}t} = \left(\gamma + v_\perp\frac{\partial\gamma}{\partial v_\perp}\right)\frac{\mathrm{d}v_\perp}{\mathrm{d}t} = \gamma\left(1 + v_\perp^2\gamma^2\right)\frac{\mathrm{d}^2\delta x_\perp}{\mathrm{d}t^2} \simeq \gamma_0\frac{\mathrm{d}^2\delta x_\perp}{\mathrm{d}t^2}\,, \tag{2.44}$$

where we have also used the fact that $v_\perp^2\gamma_0^2 \ll 1$. This is not an additional assumption, but rather it is automatically satisfied at strong coupling, in view of the Langevin dynamics and of the limitation (2.11) on the velocity of the heavy quark. Indeed, as we shall verify in section 4.1 (see eq. (4.2) there), the balance between fluctuations and dissipation in eq. (2.39) is such that the heavy quark acquires a typical transverse momentum $p_\perp$ with $(p_\perp/M_Q)^2 \lesssim \sqrt{\gamma}T/M_Q$. Then we can write

$$v_\perp^2\gamma^2 = \frac{p_\perp^2}{M_Q^2} \lesssim \frac{\sqrt{\gamma}T}{M_Q} \lesssim \frac{1}{\sqrt{\lambda}} \ll 1\,, \tag{2.45}$$

where we have also used the inequality (2.11) and the fact that the coupling is strong.

To summarize, to the accuracy of interest, we have derived the following Langevin equations for the dynamics of the heavy quark

$$\frac{\mathrm{d}p_\perp^i}{\mathrm{d}t} = -\eta_D p_\perp^i + \xi_\perp^i(t), \qquad \langle\xi_\perp^i(t)\xi_\perp^j(t')\rangle = \kappa_\perp\delta^{ij}\delta(t - t')\,, \tag{2.46}$$

$$\frac{\mathrm{d}p_\ell}{\mathrm{d}t} = -\eta_D p_\ell + \xi_\ell(t), \qquad \langle\xi_\ell(t)\xi_\ell(t')\rangle = \kappa_\ell\delta(t - t')\,, \tag{2.47}$$

where the upper index $i = 1$, 2 in eq. (2.46) distinguishes between the two possible transverse directions, $\kappa_\perp$ and κ_ℓ are given in eq. (2.38), and

$$\eta_D \equiv \frac{\eta}{M_Q} = \frac{\pi\sqrt{\lambda}}{2M_Q}T^2\,. \tag{2.48}$$

The general structure of these equations — with a friction term (or 'drag force') describing dissipation and a noise term describing momentum broadening — is as expected, and so

are the above expressions for η_D, $\kappa_\perp$ and κ_ℓ, which agree with previous calculations in the literature [5–7]. It is however important to keep in mind that eqs. (2.46)–(2.48) have been derived here only for the situation where the fluctuations in the velocity of the heavy quark remain small as compared to its average velocity v_0. To ensure that this is indeed the case, eq. (2.47) for the longitudinal motion must be supplemented with an external force which is tuned to reproduce the average motion. (Without such a term, eq. (2.47) would describe the rapid deceleration of the heavy quark due to its interactions in the plasma. Such a deceleration may entail additional phenomena, like bremsstrahlung, which are not encoded in the above equations; see the discussion in refs. [14, 19, 24].) Namely, we shall add to the r.h.s. of eq. (2.47) a term $F_{\text{ext}} = \eta\gamma_0 v_0$ which for large times equilibrates the average friction force and thus enforces a constant average velocity v_0. Further consequences of these equations will be discussed in section 4.

3   Bulk picture of the stochastic motion

In the previous section we have obtained a set of Langevin equations for the heavy quark by integrating out the fluctuations of the upper part of the string, from the world-sheet horizon up to the boundary. The noise terms in these equations have been generated via boundary conditions at the world-sheet horizon, cf. eq. (2.20). This suggests that, within the context of the supergravity calculation, quantum fluctuations are somehow encoded in the world-sheet horizon. To make this more explicit, we shall follow refs. [8, 9] and construct a set of equations describing the stochastic dynamics of the upper part of the string, in which the noise term is acting on a point on the string which is infinitesimally close to the world-sheet horizon.

More precisely, we introduce a 'stretched' horizon at $z_h \equiv z_s - \epsilon$ and integrate out the fluctuations of the part of string lying between z_s and z_h. The procedure is quite similar to the one described in the previous section except that one has to fix the boundary values for the fluctuations also on the stretched horizon, rather than just on the Minkowski boundary. Denoting the respective values by ψ^h, where as before ψ stands generically for either δx_ℓ or $\delta x_\perp$, this procedure yields an effective action S^h_{eff} for ψ^h with the same formal structure as exhibited in eq. (2.25), that is,

$$iS^h_{\text{eff}} = -i \int \frac{d\omega}{2\pi} \psi^h_a(-\omega) G^h_R(\omega) \psi^h_r(\omega) - \frac{1}{2} \int \frac{d\omega}{2\pi} \psi^h_a(-\omega) G^h_{\text{sym}}(\omega) \psi^h_a(\omega). \qquad (3.1)$$

(We temporarily omit the term linear in the fluctuations; this will be restored in the final equations.) The horizon Green's functions G^h_R and G^h_{sym} will be shortly constructed. The calculations being quite involved, it is convenient to start with a brief summary of our main results.

The r-fields $\psi_r(\omega, z)$ describing the string fluctuations within the bulk ($z_m \leq z \leq z_h$) obey the equations of motion (2.7) with Neumann boundary conditions at $z = z_m$ — meaning that the string endpoint on the boundary is freely moving (except for the imposed longitudinal motion with velocity v_0) — and with Dirichlet boundary conditions at $z = z_h$: $\psi_r(\omega, z_h) = \psi^h_r(\omega)$. This boundary field $\psi^h_r(\omega)$ is however a stochastic variable,

whose dynamics is described by the effective action (3.1). Via the classical solutions, this stochasticity is transmitted to the upper endpoint of the string, i.e., to the heavy quark. As a result, the latter obeys the same Langevin equations as previously derived in section 2.

We start with the partition function encoding the quantum dynamics of the upper part of the string ($z_m \leq z \leq z_h$) in the Gaussian approximation:

$$Z = \int \left[D\psi_R^0 D\psi_R D\psi_R^h \right] \left[D\psi_L^0 D\psi_L D\psi_L^h \right] e^{iS_R - iS_L + iS_{\text{eff}}^h}. \tag{3.2}$$

The different measures $D\psi^0$, $D\psi$ and $D\psi^h$ correspond, respectively, to the path integral over the string endpoint on the Minkowski boundary, over the bulk part of the string, and over the point of the string on the stretched horizon (separately for the left and right quadrants of the Kruskal plane). Also, S_R and S_L are defined as in eq. (2.4), but with the integral over z restricted to $z_m < z < z_h$. In what follows we shall construct the various pieces of the action which enter the exponent in eq. (3.2).

(I) The effective action at the stretched horizon, S_{eff}^h. As anticipated, this is obtained by integrating out the string fluctuations within the infinitesimal strip $z_h < z < z_s$. To that aim, we need the classical solutions $\psi_R(\omega, z)$ and $\psi_L(\omega, z)$ in the Kruskal plane which take the boundary values $\psi_R^h(\omega)$ and $\psi_L^h(\omega)$ at $z = z_h$ and are related by the condition (2.20). Clearly, the respective solutions read (in the (r, a) basis, for convenience)

$$\psi_r(\omega, z) = \psi_r^h(\omega)\psi_{\text{ret}}^h(\omega, z) + \frac{1 + 2n(\omega)}{2} \psi_a^h(\omega)(\psi_{\text{ret}}^h(\omega, z) - \psi_{\text{adv}}^h(\omega, z)),$$
$$\psi_a(\omega, z) = \psi_a^h(\omega)\psi_{\text{adv}}^h(\omega, z), \tag{3.3}$$

where ψ_{ret}^h and ψ_{adv}^h are rescaled versions of the retarded and advanced solutions introduced in section 2.2 which are normalized to 1 at $z = z_h$; e.g., $\psi_{\text{ret}}^h(\omega, z) = \psi_{\text{ret}}(\omega, z)/\psi_{\text{ret}}(\omega, z_h)$. For z close to z_s (and hence to z_h as well), these functions can be expanded as

$$\psi_{\text{ret}}^h(\omega, z) = 1 + O(z_s - z),$$
$$\psi_{\text{adv}}^h(\omega, z) = \left(\frac{z_s - z}{z_s - z_h} \right)^{\frac{i\omega z_H}{2z_s}} \left[1 + \mathcal{O}(z_s - z) \right]. \tag{3.4}$$

Substituting these classical solutions into the action produces the boundary action shown in eq. (3.1), with G_R^h defined by the horizon version of eq. (2.26). Given the near-horizon behaviour of the solutions (3.4) and of the local tension $T^{zz}(z)$ (which vanishes at $z = z_s$, cf. eq. (2.6)), it is clear that only $\partial_z \psi_{\text{adv}}^h$ contributes to G_R^h in the limit $\epsilon \to 0$. This yields the following, purely imaginary, result:

$$G_{\perp, R}^h(\omega) = z_s^4 G_{\ell, R}^h(\omega) = -\frac{1}{2} T_\perp^{zz}(z_h)\partial_z \psi_{\text{adv}}^h(-\omega, z)\big|_{z=z_h} = -i\omega\gamma\eta. \tag{3.5}$$

This coincides, as it should, with the imaginary part of the respective boundary propagator,[3] eq. (2.31) (cf. the discussion at the end of section 2.2). Then the symmetric Green's

---

[3]Incidentally, this calculation of $\text{Im} \, G_R^h$, which is exact, together with the conservation law $\partial_z J^z = 0$, cf. eq. (2.34), can be used to check, or even derive, the expressions for $\text{Im} \, C_{\text{ret}}(\omega)$ and $\text{Im} \, C_{\text{adv}}(\omega)$ given in eqs. (2.16)–(2.17).

function G^h_{sym}, which is related to $\text{Im}\, G^h_R$ via the KMS relation (2.27), is exactly the same as the corresponding function on the boundary.

If the string point on the stretched horizon was a free endpoint endowed with the action (3.1), it would obey Langevin equations similar to those derived in section 2.3. However, this is an internal point on the string, and as such it is also subjected to a tension force from the upper side of the string at $z < z_h$. This force is encoded in the bulk action $S_R - S_L$, to which we now turn.

(II) The bulk piece of the action $S_R - S_L$. This is defined as

$$
\begin{aligned}
S_R - S_L &= -\frac{1}{2} \int_{z_m}^{z_h} \mathrm{d}z \int \mathrm{d}t\, T^{\alpha\beta}(z) \Big[\partial_\alpha \psi_R \partial_\beta \psi_R - \partial_\alpha \psi_L \partial_\beta \psi_L \Big] \\
&= \frac{1}{2} \int_{z_m}^{z_h} \mathrm{d}z \int \mathrm{d}t \left[\psi_R \partial_\alpha \Big(T^{\alpha\beta} \partial_\beta \psi_R \Big) - \psi_L \partial_\alpha \Big(T^{\alpha\beta} \partial_\beta \psi_L \Big) \right] \\
&\quad - \frac{1}{2} \int \mathrm{d}t\, T^{zz}(z) \Big(\psi_R \partial_z \psi_R - \psi_L \partial_z \psi_L \Big) \Big|_{z=z_m}^{z=z_h}
\end{aligned}
\tag{3.6}
$$

or, after going to Fourier space and to the (r, a)-basis,

$$
\begin{aligned}
S_R - S_L &= \int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}z\, \psi_a(-\omega, z) \partial_\alpha \left[T^{\alpha\beta}(z) \partial_\beta \psi_r(\omega, z) \right] \\
&\quad - \frac{1}{2} \int \frac{\mathrm{d}\omega}{2\pi} T^{zz}(z) \Big(\psi_a(-\omega, z) \partial_z \psi_r(\omega, z) + (r \leftrightarrow a) \Big) \Big|_{z=z_m}^{z=z_h}.
\end{aligned}
\tag{3.7}
$$

We were so explicit here about the integration by parts, because this operation turns out to be quite subtle. First, notice that the contributions proportional to T^{zt} have cancelled in the boundary terms, for the same reason as discussed below eq. (2.12), i.e., because they are odd functions of t (or ω). To ensure this property, it has been important to perform the previous operations in the order indicated above, that is, to first integrate by parts, as in eq. (3.6), and only then change to the (r, a)-basis, as in eq. (3.7). (Reversing this order would have affected the symmetry properties of the integrand, and then the terms $\propto T^{zt}$ would not cancel anymore.)

Second, there are some subtleties about the boundary value of $S_R - S_L$ at the stretched horizon, that we rewrite here for more clarity:

$$
(S_R - S_L)^h_{\text{bndry}} = -\frac{1}{2} \int \frac{\mathrm{d}\omega}{2\pi} T^{zz}(z_h) \Big(\psi_a(-\omega, z) \partial_z \psi_r(\omega, z) + (r \leftrightarrow a) \Big) \Big|_{z=z_h}
\tag{3.8}
$$

If we were to evaluate this action with the classical solutions (3.3), the result would precisely cancel the effective action (3.1) in the exponent of eq. (3.2). Indeed, up to a sign, eq. (3.8) has exactly the structure that has been used to build the effective action by inserting the classical solutions (compare to eq. (2.24)). However, in the present context, eq. (3.8) must be rather seen as the boundary value of the bulk action when approaching the stretched horizon from the *above* (i.e., from $z < z_h$), and as such it provides boundary conditions for the dynamics of the upper side of the string (see below). This being said, it is nevertheless possible, and also convenient, to use the appropriate piece of eq. (3.8) in order to cancel

the dissipative piece $\propto G_R^h$ in the effective action (3.1). This is simply the statement that the totality of the energy which crosses the stretched horizon coming from the above flows further down along the string.

Specifically, the relevant piece of eq. (3.8) is that proportional to $\partial_z \psi_a$, which after using eq. (3.3) can be evaluated as

$$-\frac{1}{2} T^{zz}(z_h) \partial_z \psi^h_{\mathrm{adv}}(-\omega, z)\big|_{z=z_h} = G_R^h(\omega) \tag{3.9}$$

where we have recognized the expression (3.5) for G_R^h. One thus obtains:

$$(S_R - S_L)^h_{\mathrm{bndry}} = -\int \frac{\mathrm{d}\omega}{2\pi} \, \psi^h_a(-\omega) \left[\frac{1}{2} \, T^{zz}(z) \partial_z \psi_r(\omega, z) - G_R^h(\omega) \psi_r^h(\omega)\right]_{z=z_h}. \tag{3.10}$$

As anticipated, the last term in eq. (3.10) compensates the piece involving G_R^h in eq. (3.1), and then the total action reads

$$iS_R - iS_L + iS^h_{\mathrm{eff}} = i \int \frac{\mathrm{d}\omega}{2\pi} \, \psi^0_a(-\omega) \left[\frac{1}{2} \, T^{zz}(z)\Big(\partial_z \psi_r(\omega, z) + \psi_r^0(\omega)\partial_z \psi_{\mathrm{adv}}(-\omega, z)\Big) - P^z\right]_{z=z_m}$$
$$+ i \int_{z_m}^{z_h} \mathrm{d}z \int \frac{\mathrm{d}\omega}{2\pi} \, \psi_a(-\omega, z) \partial_\alpha \left[T^{\alpha\beta}(z)\partial_\beta \psi_r(\omega, z)\right]$$
$$- i \int \frac{\mathrm{d}\omega}{2\pi} \, \psi^h_a(-\omega) \left[\frac{1}{2} \, T^{zz}(z)\partial_z \psi_r(\omega, z) - \xi^h(\omega)\right]_{z=z_h} \tag{3.11}$$

where the differences between longitudinal and transverse fluctuations are kept implicit (in particular, it is understood that the term proportional to P^z appears only in the longitudinal sector). Two additional manipulations have been necessary to write the action in its above form: (i) In the first line of eq. (3.11) we have used $\psi_a(\omega, z) = \psi^0_a(\omega)\psi_{\mathrm{adv}}(\omega, z)$, cf. eq. (2.23). (ii) The piece involving $G^h_{\mathrm{sym}}(\omega)$ in eq. (3.1) have been reexpressed as a Gaussian path integral over the noise variables ξ^h, which therefore obey

$$\langle \xi^h(\omega)\xi^h(\omega')\rangle = 2\pi\delta(\omega + \omega')(1 + 2n)\omega\gamma\eta, \quad \text{with} \quad n(\omega) = \frac{1}{\mathrm{e}^{\omega/z_s T} - 1}. \tag{3.12}$$

Once again, eq. (3.12) involves the effective temperature $T_{\mathrm{eff}} = z_s T$.

By integrating over the fluctuating fields $\psi^0_a(-\omega)$, $\psi_a(-\omega, z)$, and $\psi^h_a(-\omega)$, we are finally left with the following set of equations of motion and boundary conditions:

1. A modified Neumann boundary condition for the string endpoint string at the boundary (we temporarily reintroduce the polarization label p with $p = \ell$ or $\perp$):

$$\frac{1}{2} \, T^{zz}_p(z) \left[\partial_z \psi^p_r(\omega, z) + \psi^{0,p}_r(\omega)\partial_z \psi_{\mathrm{adv}}(-\omega, z)\right]_{z=z_m} = \eta\gamma v \, \delta_{p\ell}. \tag{3.13}$$

2. The standard equations of motion for the fluctuations $\psi_r(\omega, z)$ of the string in the bulk at $z_m < z < z_h$ (cf. eq. (2.7)).

3. A stochastic equation for the point of the string on the stretched horizon:

$$\frac{1}{2} \, T^{zz}(z)\partial_z \psi_r(\omega, z)\big|_{z=z_h} = \xi^h(\omega). \tag{3.14}$$

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We now analyze the consequences of these equations and, in particular, emphasize the differences w.r.t. the corresponding analysis in ref. [9].

Eq. (3.14) is a Langevin equation of a special type: the noise term is precisely compensating the pulling force $T^{zz}(z_h)\partial_z\psi_r$ due to the string tension. By taking the expectation value of this equation and recalling that $T_p^{zz}(z_h) \sim \epsilon$ vanishes in the limit $\epsilon \to 0$, we conclude that $\partial_z\langle\psi_r^p(\omega,z)\rangle$ is regular near the world-sheet horizon; this implies that the average value of the classical solution is proportional to $\psi_{\rm ret}$:

$$\langle\psi_r(\omega,z)\rangle = \langle\psi_r^0(\omega)\rangle\,\psi_{\rm ret}(\omega,z)\,. \tag{3.15}$$

The normalization is fixed by the expectation value of $\psi_r^0(\omega)$ — the boundary value of $\psi_r(\omega,z)$ at $z = z_m \ll 1$.

We will now construct the solution $\psi_r(\omega,z)$ to the EOM (2.7) by specifying its boundary values, $\psi_r^0(\omega)$ and $\psi_r^h(\omega)$, at the points $z = z_m$ and $z = z_h$, respectively. After simple algebra, the respective solution can be written as

$$\psi_r(\omega,z) = \psi_r^0(\omega)\psi_{\rm ret}(\omega,z) + \left[\psi_r^h(\omega) - \psi_r^0(\omega)\psi_{\rm ret}(\omega,z_h)\right]\frac{\psi_{\rm ret}(\omega,z) - \psi_{\rm adv}(\omega,z)}{\psi_{\rm ret}(\omega,z_h) - \psi_{\rm adv}(\omega,z_h)}\,. \tag{3.16}$$

The reason why this particular writing is natural is as follows: when taking the expectation value according to eq. (3.15), we find $\langle\psi_r^h(\omega)\rangle = \langle\psi_r^0(\omega)\rangle\,\psi_{\rm ret}(\omega,z_h)$, which shows that the coefficient $\psi_r^h(\omega) - \psi_r^0(\omega)\psi_{\rm ret}(\omega,z_h)$ in front of the second term in eq. (3.16) is a random variable with zero expectation value. Clearly, this term plays the role of a noise. The statistics of this noise is determined by the horizon Langevin equation (3.14), and in turn it implies a boundary Langevin equation for $\psi_r^0(\omega)$, via the condition (3.13). Let's see how all that works in detail. We will first rewrite eq. (3.16) as

$$\psi_r(\omega,z) = \psi_r^0(\omega)\psi_{\rm ret}(\omega,z) + i\,\xi^0(\omega)\frac{\psi_{\rm ret}(\omega,z) - \psi_{\rm adv}(\omega,z)}{\operatorname{Im}G_R(\omega)}\,, \tag{3.17}$$

thus fixing the normalization of the noise term $\xi^0(\omega)$. After inserting eq. (3.17) in the Neumann boundary condition (3.13) (say, in the transverse sector), one finds[4]

$$G_R^0(\omega)\,\psi_r^0(\omega) = \xi^0(\omega)\,, \tag{3.18}$$

which is the standard form of a Langevin equation (compare to eq. (2.37)).

It remains to check that the statistics of ξ^0, as inferred from eq. (3.14), is indeed the same as previously derived in section 2.3. To that aim, we insert the form (3.17) of the solution into eq. (3.14); as already explained, the regular piece of the solution $\propto \psi_{\rm ret}(\omega,z)$ does not contribute in the limit $\epsilon \to 0$, so we are left with

$$-\frac{i}{2}\frac{\xi^0(\omega)}{\operatorname{Im}G_R(\omega)}\,T^{zz}(z)\partial_z\psi_{\rm adv}(\omega,z)\big|_{z=z_h} = \xi^h(\omega). \tag{3.19}$$

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[4]The following identities, which can be checked from eq. (2.26), are useful in this respect:

$$G_{\perp,R}^0(\omega) = -\frac{1}{2}T_\perp^{zz}(z_m)\left[\partial_z\psi_{\rm ret}(\omega,z) + \partial_z\psi_{\rm adv}(-\omega,z)\right]_{z=z_m}.$$

$$\operatorname{Im}G_R(\omega) = \frac{i}{2}T_\perp^{zz}(z_m)\,\partial_z\left[\psi_{\rm ret}(\omega,z) - \psi_{\rm adv}(\omega,z)\right]_{z=z_m}.$$

Using $T^{zz}(z)\partial_z\psi_{\mathrm{adv}}(\omega,z)|_{z=z_h} = -2i\omega\gamma\eta\psi_{\mathrm{adv}}(\omega,z_h)$, cf. eq. (3.9), together with $\operatorname{Im} G_R(\omega) = -\omega\gamma\eta$, this finally becomes

$$\psi_{\mathrm{adv}}(\omega,z_h)\,\xi^0(\omega) \,=\, \xi^h(\omega)\,. \tag{3.20}$$

This relation is in fact natural, as we argue now: from the transformation connecting ξ to ψ_a, or directly by comparing eq. (3.17) with the standard expression (2.23) for ψ_r, one can see that the strength of the noise term scales like $\xi(\omega) \sim \psi_a(\omega)\,G_{\mathrm{sym}}(\omega)$. On the other hand, eq. (2.23) implies $\psi_a^h(\omega) = \psi_{\mathrm{adv}}(\omega,z_h)\psi_a^0(\omega)$. Hence one can write

$$\xi^h(\omega) \,\sim\, \psi_a^h(\omega)\,G_{\mathrm{sym}}^h(\omega) \,\simeq\, \psi_{\mathrm{adv}}(\omega,z_h)\psi_a^0(\omega)\,G_{\mathrm{sym}}(\omega) \,\sim\, \psi_{\mathrm{adv}}(\omega,z_h)\,\xi^0(\omega)\,. \tag{3.21}$$

Remarkably, the relative factor between ξ^0 and ξ^h in eq. (3.20) does not spoil the normalization of the noise-noise correlator, because $|\psi_{\mathrm{adv}}(\omega,z_h)| = 1$, as we now demonstrate. To that aim we rely on the observation at the end of section 2.2 that the r.h.s. of eq. (2.33) is independent of z. Clearly, this remains true after replacing $\psi_{\mathrm{ret}} \to \psi_{\mathrm{adv}}$ in eq. (2.33). (Indeed, the current (2.34) is conserved for an arbitrary solution $\Psi_{\mathrm{sol}}(t,z)$.) Writing $\psi_{\mathrm{adv}}(\omega,z) = C(\omega)\psi_{\mathrm{adv}}^h(\omega,z)$, so that $\psi_{\mathrm{adv}}(\omega,z_h) = C(\omega)$, and evaluating the r.h.s. of eq. (2.33) separately at $z = z_m$ and $z = z_h$, one deduces that $|C(\omega)| = 1$, as anticipated. Thus, the 2-point function $\langle\xi^0(\omega)\xi^0(\omega')\rangle$ is indeed the same as in section 2.3. Note also that our previous argument is independent of the precise value of ϵ (the distance between the world-sheet and the stretched horizons), so long as ϵ is small enough for the near-horizon expansions to make sense. This strongly suggests that the strength of the noise remains constant along the string, from the stretched horizon up to the boundary.

4   Discussion and physical picture

In this section, we will first discuss some consequences of the previously derived Langevin equations, which support the idea that the noise terms in these equations are of non-thermal nature, and then propose a physical picture in which these fluctuations are interpreted as quantum mechanical fluctuations associated with medium-induced radiation.

4.1   Momentum distributions from the Langevin equations

An important property of the Langevin equations (2.46)–(2.47) that we would like to emphasize is that, except in the non-relativistic limit $\gamma \simeq 1$, these equations do not describe the thermalization of the heavy quark. There are several arguments to support this conclusion. For instance, in thermal equilibrium the momentum distributions should be isotropic, but this is clearly not the case for the large-time distributions generated by eqs. (2.46)–(2.47), because of the mismatch between κ_ℓ and $\kappa_\perp$ when $\gamma > 1$. Besides, in order to generate the canonical distribution for a relativistic particle, $P \propto \exp\{-\sqrt{\mathbf{p}^2 + M_Q^2}/T\}$, the noise correlations must not only be isotropic, but also obey the relativistic version of the Einstein relation, which reads $\kappa = 2ET\eta_D$ [26]. Using eqs. (2.38) and (2.48), it is easily seen that this condition is not satisfied for either transverse, or longitudinal, fluctuations (except if $\gamma = 1$, once again).

The Einstein relation is a particular form of the fluctuation-dissipation theorem, so its failure might look surprising given that the Green's functions at the basis of our Langevin equations obey the KMS condition (2.27). Recall, however, that this peculiar KMS condition involves an effective temperature $T_{\text{eff}} = T/\gamma^{1/2}$; and indeed, in the transverse sector at least, the Einstein relation appears to be formally satisfied, but with $T \to T_{\text{eff}}$. But this does not hold in the longitudinal sector, where κ_ℓ involves an additional factor γ^2. Hence, the present equations cannot lead to thermal distributions.

It is then interesting to compute the actual momentum distributions generated by these Langevin equations at large times. Consider first the transverse sector, and introduce the probability distribution $P(\boldsymbol{p}_\perp, t)$ for the transverse momentum $\boldsymbol{p}_\perp = (p_1, p_2)$ at time t:

$$P(\boldsymbol{p}_\perp, t) \equiv \int [D\xi_i] \, \delta\Big(\boldsymbol{p}_\perp - \boldsymbol{p}_\perp[\xi_i](t)\Big) \, \mathrm{e}^{-\frac{1}{2\kappa_\perp} \int \mathrm{d}t \, \xi_i(t) \xi_i(t)} \, . \tag{4.1}$$

Here, $\boldsymbol{p}_\perp[\xi_i](t)$ is the solution to eq. (2.46) corresponding to a given realization of the noise,[5] and reads (with $i = 1, 2$; we assume $p_i(0) = 0$ so that $\langle \boldsymbol{p}_\perp(t) \rangle = 0$ at any time)

$$p_i(t) = \int_0^t \mathrm{d}t' \, \mathrm{e}^{-\eta_D(t-t')} \, \xi_i(t') \, . \tag{4.2}$$

This implies $\langle p_1^2 \rangle = \langle p_2^2 \rangle \equiv \langle p_\perp^2 \rangle$ with

$$\langle p_\perp^2(t) \rangle = \frac{\kappa_\perp}{2\eta_D} \left(1 - \mathrm{e}^{-2\eta_D t}\right) \simeq \frac{\kappa_\perp}{2\eta_D} \, , \tag{4.3}$$

where the last, approximate equality holds for large times $\eta_D t \gg 1$. Returning to eq. (4.1), this gives

$$P(\boldsymbol{p}_\perp, t) = \frac{1}{2\pi \langle p_\perp^2(t) \rangle} \, \exp\left\{-\frac{p_1^2 + p_2^2}{2\langle p_\perp^2(t) \rangle}\right\} . \tag{4.4}$$

A similar expression holds in the longitudinal sector, but only after subtracting away the global motion with velocity v_0, which one can do by writing $\delta p_\ell \equiv p_\ell - p_0$ with $p_0 = M_Q \gamma_0 v_0$.

For large times $t \gg 1/\eta_D$, the transverse and longitudinal momentum distributions for the heavy quark approach the following, stationary, forms

$$P(\boldsymbol{p}_\perp) \simeq \frac{1}{2\pi \gamma^{1/2} T M_Q} \, \exp\left\{-\frac{p_1^2 + p_2^2}{2\gamma^{1/2} T M_Q}\right\} ,$$
$$P(\delta p_\ell) \simeq \frac{1}{\sqrt{2\pi \gamma^{5/2} T M_Q}} \exp\left\{-\frac{\delta p_\ell^2}{2\gamma^{5/2} T M_Q}\right\} , \tag{4.5}$$

which formally look like thermal, Maxwell-Boltzmann, distributions for *non-relativistic* particles, but with different temperatures in the transverse and longitudinal sector — $T_\perp = \gamma^{1/2} T$ and $T_\ell = \gamma^{5/2} T$ — none of them equal to the plasma temperature T. Although

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[5]In general, this solution will depend on our prescription for discretizing the time axis; this is so since the noise-noise correlator depends itself on the momentum ('multiplicative noise'). But to the accuracy of interest, we can treat γ in eq. (2.38) as the fixed quantity γ_0, and then one can safely use continuous notations.

these distributions have not been generated here through a genuine thermal dynamics, they nevertheless look thermal because they originate from the balance between fluctuations and dissipation. Note that the dependence upon the strong coupling λ has disappeared in this balance — the coupling cancelled out in the ratio $\kappa_\perp/\eta_D$. The mismatch between the effective temperatures in the longitudinal and, respectively, transverse sector reflects the difference between the importance of the respective fluctuations in the dynamics of radiation by a relativistic particle: as explained in section 2.3, the fluctuations in the longitudinal momentum associated with a given change in the velocity are amplified by an additional factor γ^2 as compared to the corresponding fluctuations in the transverse momentum. But in spite of this large amplification factor, the longitudinal fluctuations remain small enough not to change the average motion of the heavy quark: indeed, the second equation (4.5) implies $\delta p_\ell^2 \sim \gamma^{5/2} T M_Q$, which within the present assumptions is very small as compared to the average energy (or longitudinal momentum) $E = \gamma_0 M_Q$:

$$\frac{\delta p_\ell^2}{E^2} \sim \frac{\sqrt{\gamma_0}\, T}{M_Q} \lesssim \frac{1}{\sqrt{\lambda}} \ll 1 \,. \tag{4.6}$$

Note also that in the non-relativistic limit $\gamma \simeq 1$, the dissymmetry between the transverse and longitudinal sectors disappears, and the two equations (4.5) then provide the expected, isotropic, thermal distribution for a classical non-relativistic particle, in agreement with the results in refs. [8, 9].

Let us finally comment on what we expect to happen if one turns off the external force, i.e. if one prepares the heavy quark in a state with initial velocity v_0 and then let it propagate through the plasma. Clearly, the present formalism would not apply in that case, but it gives us some physical guidance about the behaviour at early times $t \lesssim 1/\eta_D \sim 1/(z_m T)$. Incidentally, this is a comparatively large period of time, since we recall that $z_m \ll 1$. Over that time, we expect the longitudinal momentum (or energy) of the quark to decrease by a factor of order 1, due to the drag force, whereas the accumulated effects of the longitudinal fluctuations will be of relative order $1/\sqrt{\lambda}$ (and even smaller for the transverse fluctuations) — that is, much smaller than the dissipation effects. We conclude that, over a relatively large period of times, the effects of fluctuations on the dynamics of the slowing-down heavy quark could be treated in some adiabatic approximation.

4.2 Physical picture: medium-induced radiation

In this section, we propose a physical picture for the dynamics of the heavy quark, as encoded in the Langevin equations (2.46)–(2.48). The general picture is that of medium-induced parton branching, as previously developed in refs. [2, 12–14], that we shall here adapt to the problem at hand. As we will see, this qualitative and admittedly crude picture provides the right parametric estimates for both the drag force and the (transverse and longitudinal) momentum broadening. Besides, it supports the non-thermal nature of the noise terms in the Langevin equations.

Due to its interactions with the strongly-coupled plasma, a heavy quark can radiate massless $\mathcal{N} = 4$ SYM quanta (gluons, adjoint scalars and fermions) which then escape in the medium, thus entailing energy loss towards the plasma and momentum broadening (due to the recoil of the heavy quark associated with successive parton emissions). This
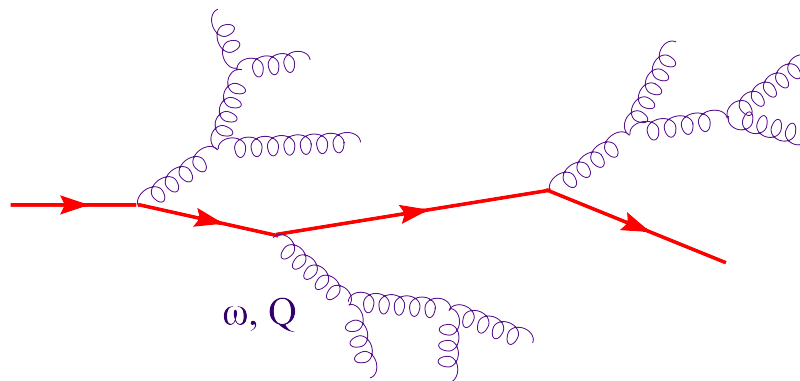
Figure 2. Energy loss and momentum broadening via medium-induced parton emission at strong coupling. It is understood that the radiated partons feel a plasma force which allows them to be liberated from the parent heavy quark (see text for details).

dynamics is illustrated in figure 2. The main ingredients underlying our physical picture are as follows:

(i) The emission of a virtual parton with energy ω and (space-like) virtuality $Q^2 = \boldsymbol{k}^2 - \omega^2 > 0$ requires a formation time $t_{\rm coh} \sim \omega/Q^2$. ($\boldsymbol{k}$ is the parton 3-momentum, and we assume high-energy kinematics: $|\boldsymbol{k}| \simeq \omega \gg Q$.) This follows from the uncertainty principle: in a comoving frame where the parton has zero momentum, its formation time is of order $1/Q$; this becomes ω/Q^2 after boosting by the parton Lorentz factor $\gamma_p = \omega/Q$. Note also that, when the parent heavy quark is highly energetic ($\gamma \gg 1$), the momentum $\boldsymbol{k}$ of the emitted parton is predominantly longitudinal: $|\boldsymbol{k}| \simeq k_\ell \simeq \omega$, whereas $k_\perp \sim Q \ll k_\ell$.

(ii) During the formation time $t_{\rm coh} \sim \omega/Q^2$, the heavy quark does not radiate just a single parton, but rather a large number of quanta, of $\mathcal{O}(\sqrt{\lambda})$, whose emissions are uncorrelated with each other. This is merely an assumption, which as we shall see provides the right λ-dependence for the final results.

(iii) Only those quanta can be lost towards the plasma, whose virtualities are small enough — smaller than the *saturation momentum* $Q_s \sim t_{\rm coh} T^2$ corresponding to the parton formation time. This follows from the analysis in ref. [12] which shows that an energetic parton propagating through the strongly coupled plasma feels the latter as a constant force $\sim T^2$ which acts towards reducing its transverse momentum (or virtuality). Then, a space-like parton, which would be stable in the vacuum, can decay inside the plasma provided the lifetime $t_{\rm coh}$ of its virtual fluctuations is large enough for the mechanical work $\sim t_{\rm coh} T^2$ done by the plasma to compensate the energy deficit $\sim Q$ of the parton. This condition amounts to $Q \lesssim Q_s$, with the upper limit given by

$$Q_s \sim t_{\rm coh} T^2 \sim (\omega T^2)^{1/3} \sim \sqrt{\gamma_p}\, T \,, \tag{4.7}$$

where we have also used $t_{\rm coh} \sim \omega/Q^2$ and $\gamma_p = \omega/Q$.

(iv) The rapidities of the radiated quanta are bounded by the rapidity of the heavy quark: $\gamma_p \lesssim \gamma$. This is again motivated by the uncertainty principle and at least at weak coupling it is confirmed by the explicit construction of the heavy quark wavefunction [27].

We shall now use this picture to compute the rate for energy loss and momentum broadening of the heavy quark. The latter radiates energy $\Delta E \sim \sqrt{\lambda}\,\omega$ over a time interval $\Delta t \sim \omega/Q^2$, where ω and Q are constrained by $Q \lesssim Q_s(\omega, T)$. The dominant contribution to the rate $|\Delta E/\Delta t|$ comes from those quanta carrying the maximal possible energy $\omega \simeq \gamma Q$ and also the maximal corresponding virtuality $Q \simeq Q_s(\gamma, T) \sim \sqrt{\gamma}\,T$ (to minimize the emission time). Therefore,

$$ -\frac{\mathrm{d}E}{\mathrm{d}t} \simeq \frac{\sqrt{\lambda}\,\omega}{(\omega/Q_s^2)} \simeq \sqrt{\lambda}\,Q_s^2 \sim \sqrt{\lambda}\,\gamma\,T^2\,, \tag{4.8} $$

in qualitative agreement with the estimate for the drag force $F_{\mathrm{drag}} = \eta\gamma v \sim \gamma\sqrt{\lambda}\,T^2$ in eq. (2.47). (Recall that we consider the relativistic case $v \simeq 1$.)

Consider similarly momentum broadening: being uncorrelated with each other, the $\sqrt{\lambda}$ quanta emitted during a time interval t_{coh} have transverse momenta which are randomly oriented, so their emission cannot change the *average* transverse momentum of the heavy quark. However, the changes in the *squared* momentum add incoherently with each other, thus yielding (once again, the dominant contribution comes from quanta with $Q \sim Q_s(\gamma, T)$ and $\omega \simeq \gamma Q$)

$$ \frac{\mathrm{d}\langle p_\perp^2 \rangle}{\mathrm{d}t} \sim \frac{\sqrt{\lambda}\,Q_s^2}{(\omega/Q_s^2)} \sim \sqrt{\lambda}\,\frac{Q_s^4}{\gamma Q_s} \sim \sqrt{\lambda}\,\sqrt{\gamma}\,T^3\,, \tag{4.9} $$

which is parametrically the same as the estimate for $\kappa_\perp$ in the first equation (2.38). The random emissions also introduce fluctuations in the energy (or longitudinal momentum) of the heavy quark, in addition to the average energy loss. The dispersion associated with such fluctuations is estimated similarly to eq. (4.9) (below, $\delta p_\ell \equiv p_\ell - \langle p_\ell \rangle$)

$$ \frac{\mathrm{d}\langle \delta p_\ell^2 \rangle}{\mathrm{d}t} \sim \frac{\sqrt{\lambda}\,\omega^2}{(\omega/Q_s^2)} \sim \sqrt{\lambda}\,\sqrt{\gamma}\,\gamma^2\,T^3\,, \tag{4.10} $$

in qualitative agreement with the previous result, eq. (2.38), for κ_ℓ. Note that, with this interpretation, the relative factor γ^2 in between κ_ℓ and $\kappa_\perp$ is simply the consequence of the relation $\omega \simeq \gamma Q$ between the energy and the virtuality (or transverse momentum) of an emitted parton.

This physical picture also clarifies the role of the world-sheet horizon in the dual gravity calculation: via the UV/IR correspondence, the radial position $z_s = 1/\sqrt{\gamma}$ of this horizon (in units of $z_H = 1/\pi T$) is mapped onto the saturation momentum $Q_s \sim \sqrt{\gamma}\,T$ in the boundary, gauge, theory. Hence the emergence of the noise terms from the near-horizon dynamics of the string reflects quantum-mechanical fluctuations in the emission of quanta with virtualities $Q \sim Q_s$, which as we have just seen control momentum broadening.
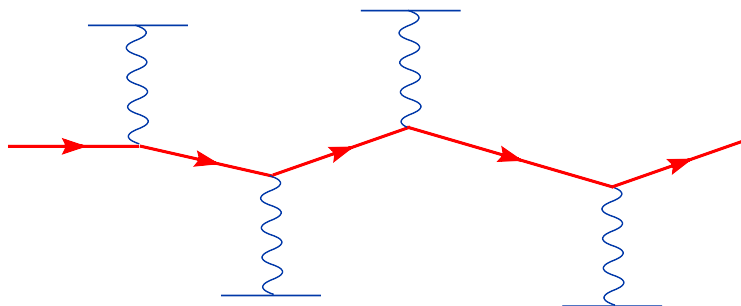
Figure 3. Momentum broadening via thermal rescattering at weak coupling.
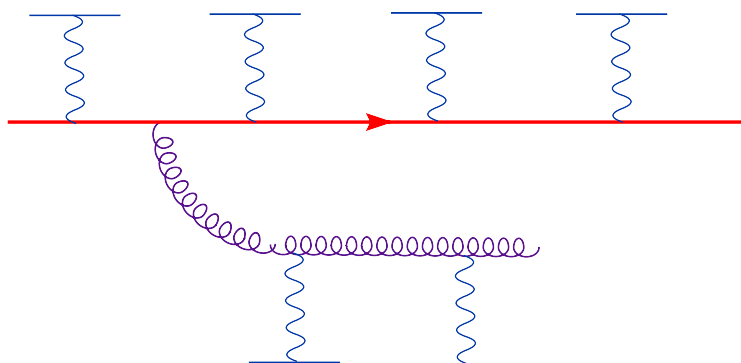

Figure 4. Energy loss via medium-induced gluon emission at weak coupling

It is furthermore interesting to compare the above physical picture to the corresponding one at weak coupling [15–18]. Note first that the mechanism for momentum broadening is different in the two cases: at weak coupling, this is dominated by thermal rescattering, i.e., by successive collisions with the plasma constituents which are thermally distributed (see figure 3). In that case, the rate $d\langle p_\perp^2 \rangle / dt \equiv \hat{q}$ defines a genuine transport coefficient — the "jet-quenching parameter" — i.e. a local quantity which depends only upon the local density of thermal constituents (quarks and gluons) together with the gluon distribution produced via their high-energy evolution. By contrast, at strong coupling, the dominant mechanism at work is medium-induced radiation, which is intrinsically non-local (it requires the formation time $t_{\rm coh}$) and hence cannot be expressed in terms of a local transport coefficient. Medium-induced radiation is of course possible at weak coupling too (see figure 4), but the respective contribution is suppressed by a factor $g^2 N_c$ as compared to the thermal rescattering. We see that, formally, it is the replacement $g^2 N_c \to \sqrt{\lambda}$ (i.e., the coherent emission of a large number of quanta) which makes the medium-induced radiation become the dominant mechanism for momentum broadening at strong coupling.

On the other hand, energy loss is predominantly due to medium-induced radiation at both weak and strong coupling, but important differences occur between the detailed mechanisms in the two cases (compare figures 2 and 4). At weak coupling, the radiated gluon, which typically comes from a highly virtual gluon in the quark wavefunction, is freed (radiated) via thermal rescattering. At strong coupling, radiation is caused by the

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plasma force $\sim T^2$. After being emitted, the parton undergoes successive medium-induced branchings, thus producing a system of partons with lower and lower energies and transverse momenta, down to values of $\mathcal{O}(T)$, when the partons cannot be distinguished anymore from the thermal bath.

It is finally interesting to notice that, in spite of such physical dissymmetry, the formula for energy loss at weak coupling can be written in a form which ressembles eq. (4.8), namely

$$- \frac{\mathrm{d}E}{\mathrm{d}t} \simeq g^2 N_c Q_s^2 \qquad \text{(weak coupling)}, \tag{4.11}$$

where however Q_s is now the saturation momentum to lowest order in perturbative QCD and is related to the respective jet-quenching parameter via $Q_s^2 \simeq \hat{q} t_{\mathrm{coh}}$. Energy loss involves a coherent phenomenon at both weak and strong coupling.

Acknowledgments

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